Dimensionles Groups

Definitions and Usage

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BIOT NUMBER

Bi

Heat Transfer to the Fluid / Heat Transfer within the Fluid

$$Bi = \frac{hl}{k}$$

- h: *convective heat transfer coefficient (W/m^2K)*
- l: *characteristic length (m)*
- k: *conductive heat transfer coefficient (W/m K)*

Modelling Application: Transient heat transfer, and to identify dominant heat transfer mode. Biot < 0.1 implies heat conduction within the body is much faster than the heat conduction away from the surface, and a lumped-capacitance model would have less than 5% error.

CAUCHY NUMBER

C

Inertia Force / Elastic Force

$$C = \frac{\rho v^2}{p}$$

- ρ: *fluid density (kg/m^3)*
- v: *fluid velocity (m/s)*
- p: *fluid pressure (Pa)*

Modelling Application: Measure of compressibility, however the Mach number is more often used. For solids pressure should be replaced by the Young's Modulus of elasticity.

CAVITATION NUMBER

Ca

Pressure Force / Inertia Force

$$Ca = \frac{p - p_v}{\frac{1}{2}\rho v^2}$$

- p: *fluid pressure (Pa)*
- p_v: *fluid vapor pressure (Pa)*
- v: *fluid velocity (m/s)*
- ρ: *fluid density (kg/m^3)*

Modelling Application: Modeling cavitation phenomenon.

EULER NUMBER

Eu

Pressure Force / Inertia Force

$$Eu = \frac{\Delta p}{\frac{1}{2}\rho v^2}$$

- Δp: *local free stream pressure (Pa)*
- v: *free stream velocity (m/s)*
- ρ: *free stream density (kg/m^3)*

Modelling Application: Fluid motion: this is half of the pressure coefficient, and is also similar to the cavitation number.

FOURIER NUMBER

Fo

$$\frac{\text{Energy Conducted}}{\text{Energy Stored}}$$

$$Fo = \frac{\alpha t}{l^2}$$

- α – *heat diffusivity (m^2/s)*
- t – *time (s)*
- l – *characteristic length (m)*

$$Fo = \frac{kt}{\rho C_p l^2}$$

k - conduction heat transfer (W/m K)
C_p - specific heat capacity (J/kg K)
ρ - density (kg/m^3)

Modelling Application: Analysis of transient heat transfer.

FROUDE NUMBER

Fr

$$\frac{\text{Inertia Force}}{\text{Gravity Force}}$$

$$Fr = \frac{v^2}{gl}$$

- v – *fluid velocity (m/s)*
- l – *characteristic length (m)*
- g – *gravitational acceleration (m/s^2)*

Modelling Application: Analysis of free surface flows (ships), and open channel flow.

GRAETZ NUMBER

Gz

$$\frac{\text{Thermal Capacity}}{\text{Conductive Heat Transfer}}$$

$$Gz = \frac{\dot{m} c_p}{kl}$$

- $\dot{m}$ – *mass flow rate (kg/s)*
- c_p – *specific heat capacity (J/kg K)*
- l – *characteristic length (m)*
- k – *conductive heat transfer coefficient (W/m K)*

Modelling Application: Conductive heat transfer.

GRASHOF NUMBER

Gr

$$\frac{\text{Buoyancy Force}}{\text{Viscosity Force}}$$

$$Gr = \frac{g \beta \Delta T l^3}{\nu^2}$$

- g – *gravitational acceleration (m/s^2)*
- β – *coefficient of cubic expansion (1/K)*
- ΔT – *temperature difference (K)*
- l – *characteristic length (m)*
- ν – *kinematic viscosity (m^2/s)*

Modelling Application: Natural convection.

LEWIS NUMBER

Le

$$\frac{\text{Mass Diffusivity}}{\text{Thermal Diffusivity}}$$

$$Le = \frac{D}{\alpha}$$

D – *mass diffusivity (m^2/s)*

α – *heat diffusivity (m^2/s)*

$$Le = \frac{D}{k/\rho C_p}$$

k - conductive heat transfer (W/m K)
C_p - specific heat capacity (J/kg K)
ρ - fluid density (kg/m^3)

Modelling Application: Study of turbulent flow, flow transitions, and diffusion.

MACH NUMBER

M

$$\sqrt{\frac{\text{Inertia Force}}{\text{Elastic Force}}}$$

$$M = \frac{v}{a}$$

v – *fluid velocity (m/s)*

a – *local speed of sound in the fluid (m/s)*

Modelling Application: Analysis of compressible fluids, transonic flow, and supersonic flow.

NUSSELT NUMBER

Nu

$$\frac{\text{Total Heat Transfer}}{\text{Conductive Heat Transfer}}$$

$$Nu = \frac{hl}{k}$$

h – *convective heat transfer coefficient (W/m^2 K)*

l – *characteristic length (m)*

k – *conductive heat transfer coefficient (W/m K)*

Modelling Application: Convective heat transfer.

PECLET NUMBER

Pe

$$\frac{\text{Bulk Mass Transfer}}{\text{Diffusive Mass Transfer}}$$

$$Pe = \frac{vl}{D}$$

v – *fluid velocity (m/s)*

l – *characteristic length (m)*

D – *mass diffusivity (m^2/s)*

Modelling Application: Diffusion and mass transfer, however if D is replaced with α this then becomes the heat convected divided by the heat conducted (i.e the Peclet heat transfer number).

PRANDTL NUMBER

Pr

$$\frac{\text{Momentum Diffusivity}}{\text{Thermal Diffusivity}}$$

$$Pr = \frac{\nu}{\alpha}$$

- ν – *kinematic viscosity (m^2/s)*
- α – *heat diffusivity (m^2/s)*

$$Pr = \frac{C_p \mu}{k}$$

k - conduction heat transfer (W/m K)
C_p - specific heat capacity (J/kg K)
μ - dynamic viscosity (Pa s)

Modelling Application: Boundary layer studies.

RAYLEIGH NUMBER

Ra

$$\frac{\text{Gravity Diffusivity}}{\text{Thermal Diffusivity}}$$

$$Ra = \frac{g \beta \Delta T \, l^3}{\nu \alpha}$$

- g – *gravitational acceleration (m/s^2)*
- β – *coefficient of cubic expansion (1/K)*
- ΔT – *temperature difference (K)*
- l – *characteristic length (m)*
- α – *heat diffusivity (m^2/s)*
- ν – *kinematic viscosity (m^2/s)*

Modelling Application: Natural convection.

REYNOLDS NUMBER

Re

$$\frac{\text{Inertia Force}}{\text{Viscosity Force}}$$

$$Re = \frac{\rho v l}{\mu}$$

- ρ – *fluid density (kg/m^3)*
- v – *fluid velocity (m/s)*
- l – *characteristic length (m)*
- μ – *dynamic viscosity (Pa s)*

Modelling Application: Studying velocity profiles and characterizing fluid flows.

SCHMIDT NUMBER

Sc

$$\frac{\text{Momentum Diffusivity}}{\text{Mass Diffusivity}}$$

$$Sc = \frac{\nu}{D}$$

- ν – *kinematic viscosity (m^2/s)*
- D – *mass diffusivity (m^2/s)*

Modelling Application: Boundary layer flow.

STANTON NUMBER

Heat Transfered to Fluid / Heat Transported by Fluid

$$St = \frac{h}{\rho v C_p}$$

- h: *convective heat transfer coefficient (W/m² K)*
- C_p: *specific heat capacity (J/kg K)*
- v: *fluid velocity (m/s)*
- ρ: *fluid density (kg/m³)*

Modelling Application: Convective heat transfer.

STEFAN NUMBER

Heat Radiated / Heat Conducted

$$Sf = \frac{\eta A_r T^4}{k A_c \,{}^{\Delta T}\!/_{\Delta l}}$$

- η: *Stefan-Boltzmann (5.670×10^{-8} J/K⁴ m² s)*
- A_r: *radiation area (m²)*
- T: *temperature (K)*
- $\Delta T/\Delta l$: *conduction temperature gradient (K/m)*
- A_c: *conduction area (m²)*
- k: *conductive heat transfer coefficient (W/m K)*

Modelling Application: Radiation heat transfer.

STOKES NUMBER

Viscosity Force / Gravity Force

$$So = \frac{\mu v}{\rho g l^2}$$

- μ: *dynamic viscosity (Pa s)*
- v: *fluid velocity (m/s)*
- l: *characteristic length (m)*
- g: *gravitational acceleration (m/s²)*
- ρ: *fluid density (kg/m³)*

Modelling Application: Study particles moving through viscous flows.

STROUHAL NUMBER

Vibration Speed / Flow Speed

$$Sr = \frac{\omega l}{v}$$

- ω: *frequency of oscillation (rad/s)*
- l: *characteristic length (m)*
- v: *fluid velocity (m/s)*

Modelling Application: Vortex phenomenon and aeroelasticity.

WEBER NUMBER

We

$$\frac{\text{Inertia Force}}{\text{Surface Tension Force}}$$

$$We = \frac{\rho v^2 l}{\sigma}$$

- ρ: *fluid density (kg/m³)*
- v: *fluid velocity (m/s)*
- l: *characteristic length (m)*
- σ: *surface tension (N/m)*

Modelling Application: Free surface studies.

THEORY

" Dimensionless numbers in many fields of engineering are collections of variables that provide order-of-magnitude estimates about the behavior of a system. They are often derived by combining coefficients from differential equations and are oftentimes a ratio between two physical quantities. "

- *S. Ted Oyama, in Membrane Science and Technology, 2011*

Fundamental Dimensions

length	**l**	m (meter)
mass	**m**	kg (kilogram)
time	**t**	s (second)
temperature	**T**	°C (degree Celcius)
current	**I**	A (Ampere)

A common dimensionless number in fluid dynamics is the Reynolds number. This ratio of inertial to viscous forces was correlated with experimental evidence to describe turbulent (Re > 2900) and laminar (Re < 2000) flow in pipes. Dimensional analysis generally enables predictability in the modeling of systems, for example enabling minature ship models to represent the motion of larger vessels.

DERIVATION

The formation of dimensionless groups is formalized as the **Buckingham Pi Theorem**, stating: If there are k variables and parameters in a problem and these contain r fundamental dimensions (e.g. m, l, t) the equation governing the system will have (k - r) dimensionless groups (Π groups). Further analysis determines if any group is useful. An excellent overvew can be found here, summary below.

1 Choose parameters likely to affect variables (e.g. the force of drag (F)).

$\frac{m}{l t^2} \rightarrow$ **F, μ, ρ, v, D**; $\frac{m}{lt} \nearrow$ (μ)

k = 5

2 Select "repeating" terms (e.g. ρ, D, v), used to balance out "non-repeating" terms (e.g. F, μ).

$\frac{m}{l^3} \rightarrow$ **ρ, D, v** $\leftarrow \frac{l}{t}$; $l \uparrow$ (D)

3 Count the number of fundamental dimensions that need to be considered.

m, l, t

r = 3

4 Calculate the number of possible Π groups, these are dimensionless groups.

k - r = 2

5 Form Π group.

$$\Pi_1 = \mu \rho^a v^b D^c$$

$$\Pi_1 = \frac{m}{lt}\left(\frac{m}{l^3}\right)^a \left(\frac{l}{t}\right)^b l^c$$

$m: 0 = 1 + a \qquad a = -1$

$l: 0 = -1 - 3a + b + c \qquad c = -1$

$t: 0 = -1 - b \qquad b = -1$

$$\Pi_1 = \mu/(\rho v l)$$

USAGE

If a system consists of small particles falling through a viscous medium, and it is characterized by requiring that the ratio between gravity and inertia forces remain constant, how would the drag associated with the system scale with respect to system size?

Hold dimensionless group ratio constant & solve for velocity ratio

$$\frac{\text{Inertia Force}}{\text{Gravity Force}} \qquad Fr = \frac{v^2}{gl}$$

$$\frac{v_1^2}{g l_1} = \frac{v_0^2}{g l_0}$$

$$\frac{v_1^2}{v_0^2} = \frac{g l_1}{g l_0} \rightarrow \frac{v_1}{v_0} = \frac{\sqrt{l_1}}{\sqrt{l_0}}$$

Create drag ratio & solve for length by plugging in velocity

$$\frac{D_1}{D_0} = \frac{6\pi\mu l_1 v_1}{6\pi\mu l_0 v_0}$$

$$\frac{D_1}{D_0} = \frac{l_1}{l_0}\frac{\sqrt{l_1}}{\sqrt{l_0}} = \frac{l_1^{3/4}}{l_0^{3/4}}$$

This ratio can now be used to scale a model for testing such that the ratio of inertia and gravity forces (Freud Number) is constant. Any length scaling must be set to the 3/4 power to represent the scaling of associated drag.

Ar	**Archimedes Number** Buoyancy, entrainment, and two-phase flow
α	**Arrhenius Number** Chemical reactions, and energy analysis
Bm	**Bingham Number** Viscoelastic, plastic, and non-Newtonian flow
Bl	**Blake Number** Capillary, and two-phase flow
Bs	**Bodenstein Number** Mass and momentum transfer
Bo	**Bond Number** Capillary flow, and surface tension (Eotvos Number, Eo)
Br	**Brinkman Number** Organic liquid flow
J	**Colburn Number** Momentum and mass diffusivity
De	**Dean Number** Non-straight line flow
E	**Eckert Number** Energy dissipation
Ga	**Galileo Number** Flow against gravity
H	**Hodgson Number** Pulsating flow
Ja	**Jakob Number** Heat transfer
Kn	**Knudsen Number** Compressible flows
Z	**Ohnesorge Number** Capillary flow, and surface tension

Sh	**Sherwood Number** Mass transfer and heat transfer
So	**Sommerfeld Number** Lubrication and bearing design
C_d	**Drag Coefficient** Fluid flow over objects
f	**Friction Factor** Fluid flow over surfaces
N_p	**Power Number** Turbomachinery
C_p	**Pressure Coefficient** Fluid flow over surfaces

Equations for these additional numbers can be found here: https://www.iist.ac.in/sites/default/files/people/numbers.html

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